CSC242: Intro to AI

Lecture 9

Propositional Inference

Factored Representation

- Splits a state into variables (factors, attributes, features, “things you know”) that can have values
- Factored states can be more or less similar (unlike atomic states)
- Can also represent uncertainty (don’t know value of some attribute)

Constraint Satisfaction Problem (CSP)

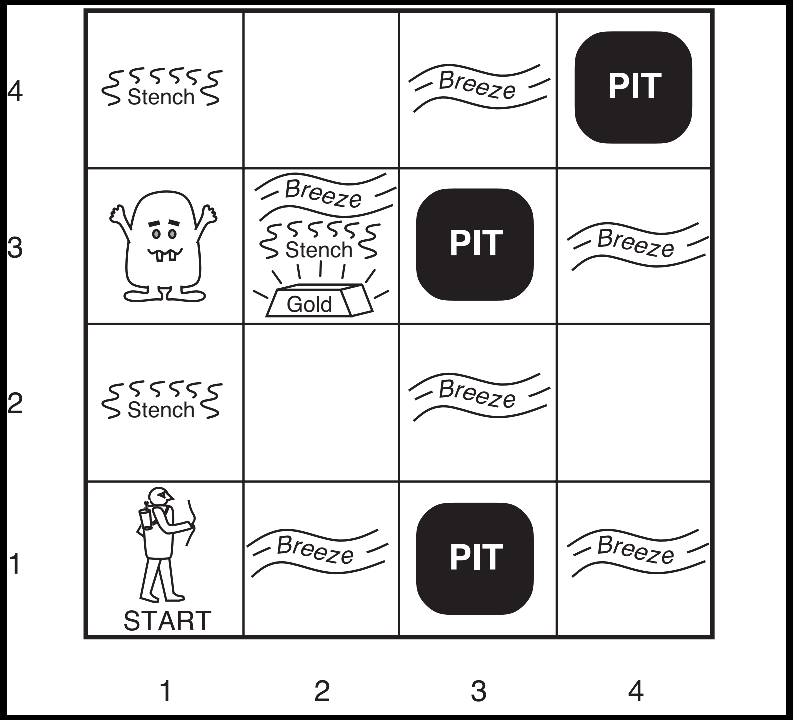
X: Set of variables $\{ X_1, ..., X_n \}$

D: Set of domains $\{ D_1, ..., D_n \}$

Each domain D_i = set of values $\{ v_1, ..., v_k \}$

C: Set of constraints $\{ C_1, ..., C_m \}$

Hunt the Wumpus



If you perceive a stench, then there the wumpus is in an adjacent square.

$$(At_{i,j} \wedge S_{i,j}) \Rightarrow (W_{i-1,j} \vee W_{i+1,j} \vee W_{i,j-1} \vee W_{i,j+1})$$

Possible Worlds

Hungry	Cranky
true	false
false	true
true	true
false	false

Possible Worlds

Hungry=true, Cranky=false

Hungry=false, Cranky=true

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Possible Worlds

Hungry=true, Cranky=false

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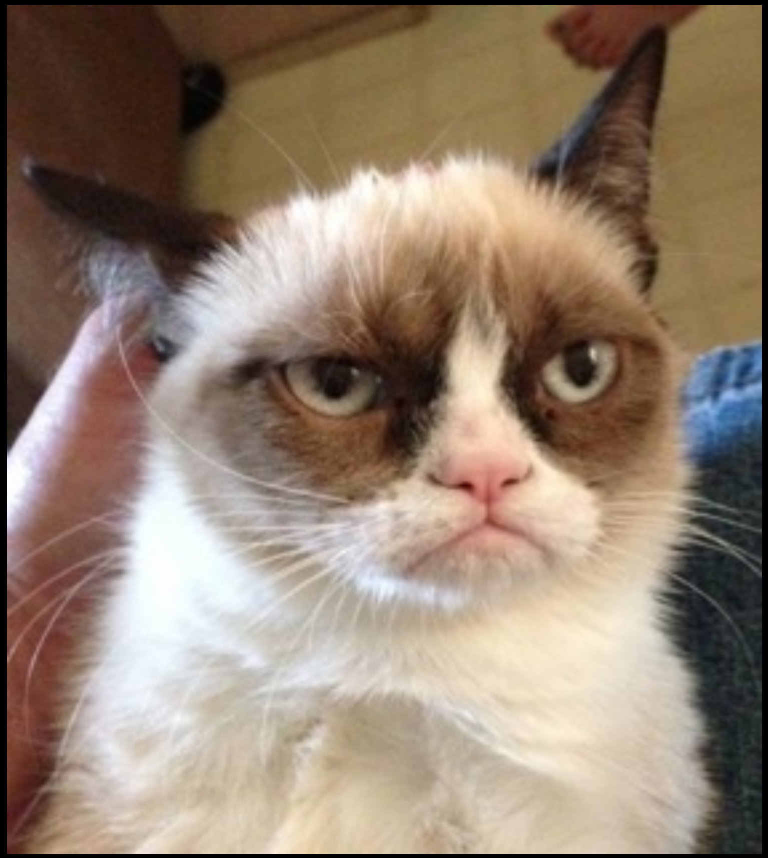
Hungry=true, Cranky=true

Hungry=false, Cranky=false

Hungry $\vee$ Cranky

Possible Worlds



Hungry=true,
Cranky=false

Hungry=false,
Cranky=true

Hungry=true,
Cranky=true

Hungry=false,
Cranky=false

Hungry ⇒ Cranky

Which Worlds are Ruled Out?

Possible Worlds

Hungry=true, Cranky=false

Hungry=false, Cranky=true

Hungry=true, Cranky=true

Hungry=false, Cranky=false

Hungry ⇒ Cranky

A sentence or set of sentences in propositional logic reduces the number of possible worlds by ruling out some of them.

- Model:
  - An assignment that satisfies a sentence or set of sentences
  - == A possible world where those sentences are true, and so is not ruled out by them
  - May be the real world! (When?)

Unsatisfiable

- A sentence or set of sentences is unsatisfiable when:
  - No complete, consistent assignment of truth values to the propositions that makes the sentence or set of sentences true
- Rules out all possible worlds
- Cannot describe the actual world

Inference

- What other things are we justified in believing, assuming our background knowledge and perceptions are accurate?
- What other sentences are true, given our background knowledge and perceptions?
- Does a given sentence or set of sentences follow from our knowledge?

What does it mean to “follow from” our knowledge?

Entailment

- α entails β when:
  - β is true in <u>every</u> world considered possible by α
  - Every model of α is also a model of β
  - Notation: $\alpha \models \beta$

Hungry ⊨ (Hungry v Cranky)

Hungry=true, Cranky=false

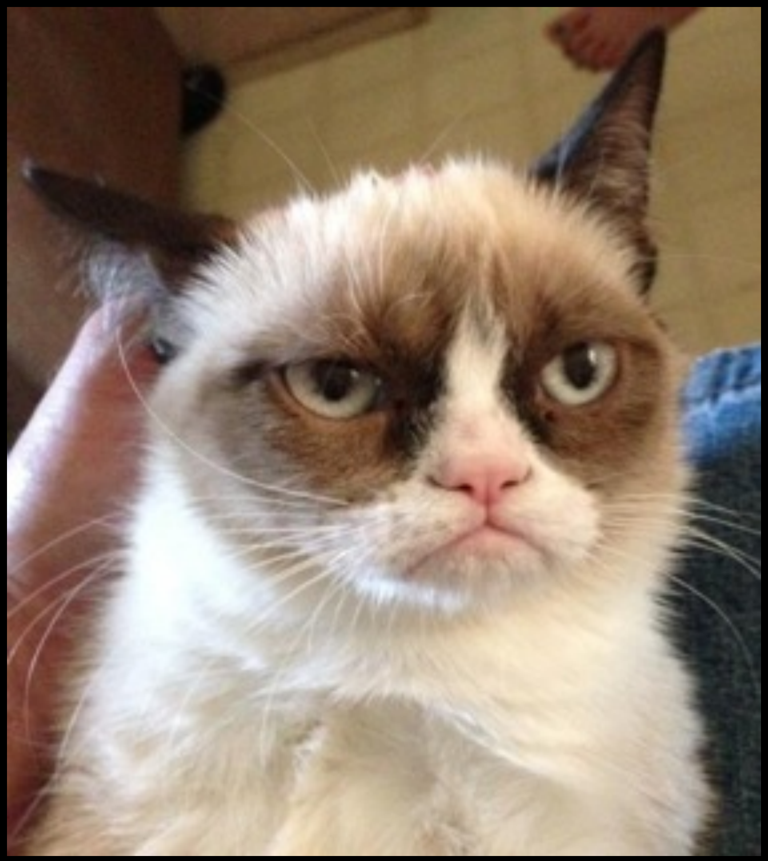
Hungry=false, Cranky=true

Hungry=true, Cranky=true

Hungry=false, Cranky=false

What are the worlds for Hungry?

Hungry ⊨ (Hungry v Cranky)



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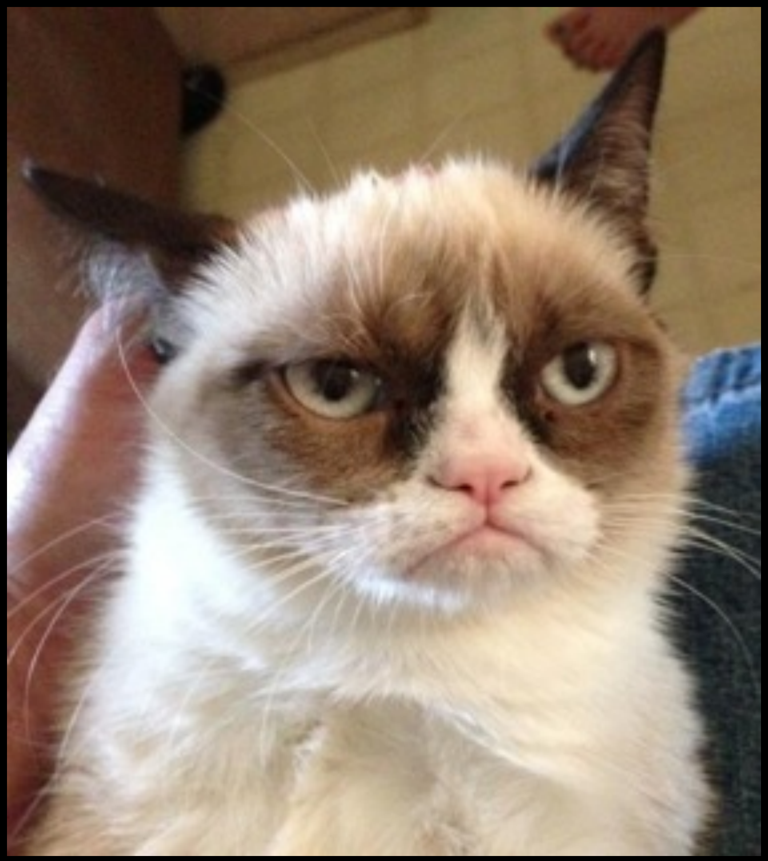

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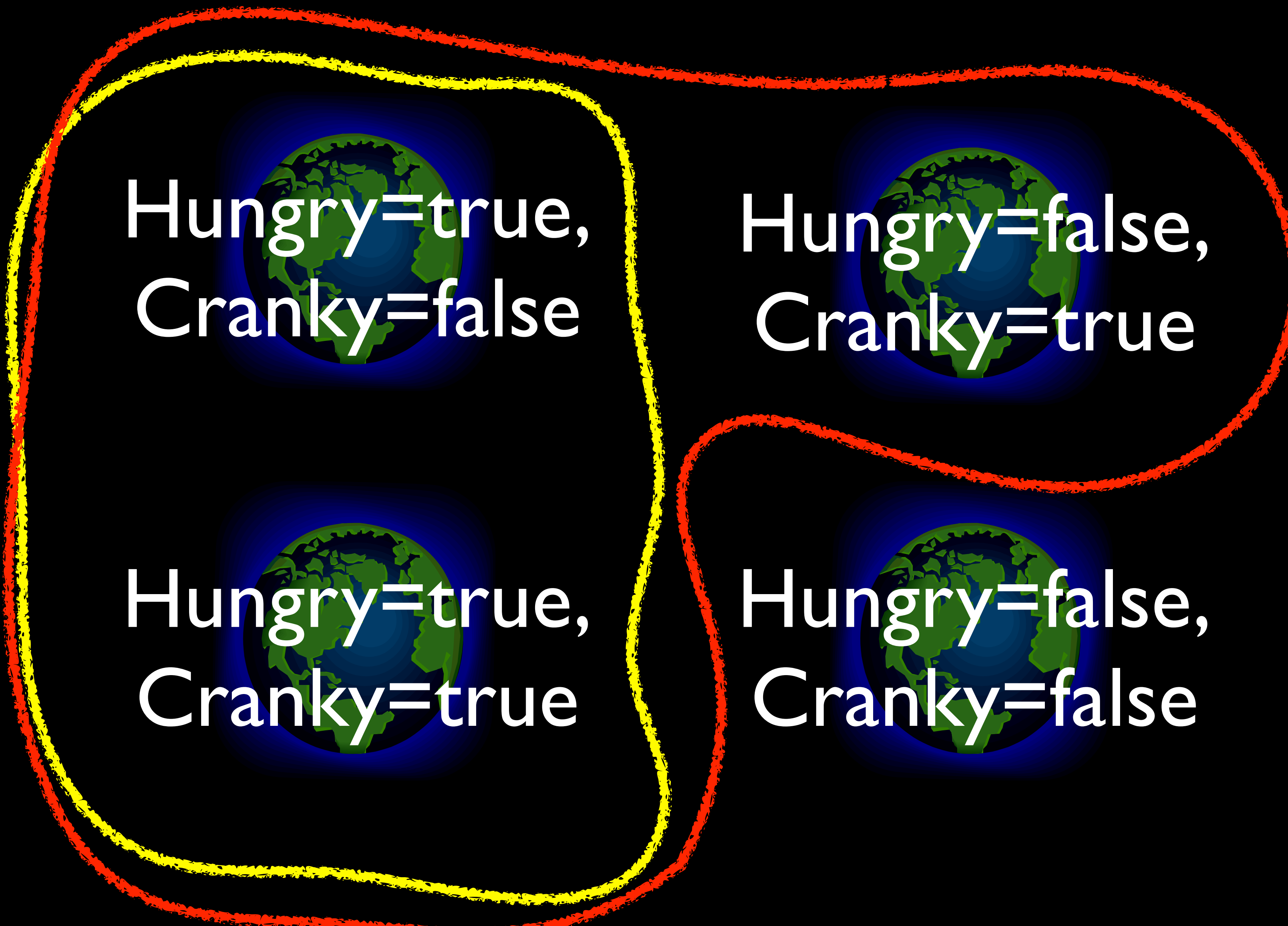
What are the worlds for Hungry?

What are the worlds for (Hungry v Cranky?)

Hungry ⊨ (Hungry v Cranky)



What are the worlds for Hungry?

What are the worlds for (Hungry v Cranky?)

Model Checking Algorithm for $\alpha \vDash \beta$

```
for every possible world W:
  if W makes α true and β false:
    return “No, α does not entail β”
return “Yes, α entails β”
```

What is the Difference Between

$\alpha \Rightarrow \beta$

$\alpha \models \beta$

What is the Difference Between

$\alpha \Rightarrow \beta$

part of sentence, like $\lor$ or $\land$

$\alpha \models \beta$

relationship between sentences

But Still…

Something must be be going on between $\alpha \Rightarrow \beta$ and $\alpha \vDash \beta$!

To get at it, we need one more concept…

Valid

- A sentence β is valid if it is true in *every* possible world
  - Every assignment is a model of β
- Example:

$P \vee \neg P$

The Connection!

- Where α and β are any two sentences,

  $\alpha \vDash \beta$ (α entails β)

  if and only if

  $\alpha \Rightarrow \beta$ is …

The Connection!

- Where α and β are any two sentences,

$$\alpha \models \beta \quad (\alpha \text{ entails } \beta)$$

if and only if

$$\alpha \Rightarrow \beta \text{ is valid}$$

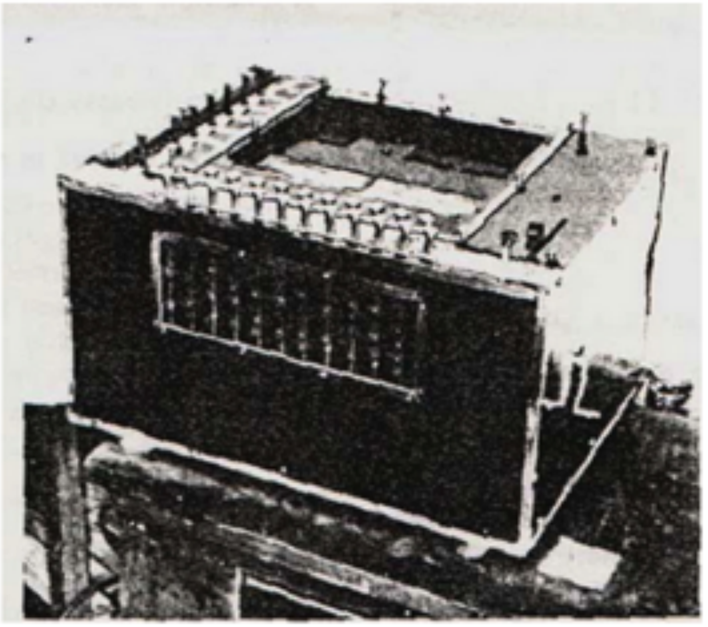
Propositional Deduction

The Problem with Model Checking

- Although we can implement propositional model checking, we wouldn't want to
- Next class, we will start learning about first-order logic, which can describe infinite sets of models!
  - Model checking is *impossible* even in *principle*

Mechanizing Reasoning

- Can we implement logical reasoning without thinking at all about possible worlds and entailment?
- The surprising answer: YES
- Logical reasoning can be performed just by syntactic manipulations on sentences!



Georges Artrouni's mechanical brain, a translation device patented in France in 1933.

Intuition: Math

Intuition: Math

$$\begin{array}{r} 123 \\ +456 \\ \hline 579 \end{array}$$

Intuition: Math

$$x + 3 = 7$$
$$x + 3 - 3 = 7 - 3$$
$$x = 7 - 3$$
$$x = 4$$

Mathematical Identities

- Allow us to rewrite equations
- Truth-preserving:
  - If the original equation holds, then so does the rewritten one

Inference Rules

- Look for rules that allow us to rewrite sentences in a truth-preserving way
- We'll call these <u>inference rules</u>, since they will allow us to do inference (draw conclusions, make implicit knowledge explicit)

$P \Rightarrow Q$

P

$\Rightarrow$

Q

$P \Rightarrow Q$

P

Q

P	Q	$P \Rightarrow Q$
true	true	true
false	true	true
true	false	false
false	false	true

$P \Rightarrow Q$

P

⇒

Q

P	Q	P⇒Q
true	true	true
false	true	true
true	false	false
false	false	true

$P \Rightarrow Q$

P

⇒

Q

P	Q	P⇒Q
true	true	true
false	true	true
true	false	false
false	false	true

$P \Rightarrow Q$

P

Q

P	Q	P⇒Q
true	true	true
false	true	true
true	false	false
false	false	true

Entailment

- α entails β when:
  - β is true in **every** world considered possible by α
  - Every model of α is also a model of β
  - *Models*(α) ⊆ *Models*(β)

$$\{ P \Rightarrow Q, P \} \models Q$$

$$\{ \alpha \Rightarrow \beta, \alpha \} \models \beta$$

Modus Ponens

$$\frac{\alpha \Rightarrow \beta, \alpha}{\beta}$$

- Modus Ponens is a sound rule of inference
- What can be derived from a set of formulas using Modus Ponens is in fact entailed by those formulas

Derivation

- β can be derived from α using inference rules
- $\alpha \vdash \beta$

Properties of Inference Rules

Soundness

- Derives only logically entailed sentences
- Truth-preserving

$$\text{if } \alpha \vdash \beta \quad \text{then } \alpha \models \beta$$

Completeness

- Derives all logically entailed sentences

if $\alpha \models \beta$ then $\alpha \vdash \beta$

Inference Rules

$$\frac{\alpha \wedge \beta}{\alpha}$$

And-elimination

$$\frac{\neg\neg\alpha}{\alpha}$$

Double negation

$$\frac{\neg(\alpha \wedge \beta)}{\neg\alpha \vee \neg\beta} \qquad \frac{\neg(\alpha \vee \beta)}{\neg\alpha \wedge \neg\beta}$$

DeMorgan's Laws

$$\frac{\alpha \Rightarrow \beta, \alpha}{\beta}$$

Modus Ponens

$$\frac{\alpha \Leftrightarrow \beta}{(\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha)} \qquad \frac{(\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha)}{\alpha \Leftrightarrow \beta}$$

Definition of biconditional

also

- Should add contrapositive rule

Want to compute whether $\alpha \models \beta$

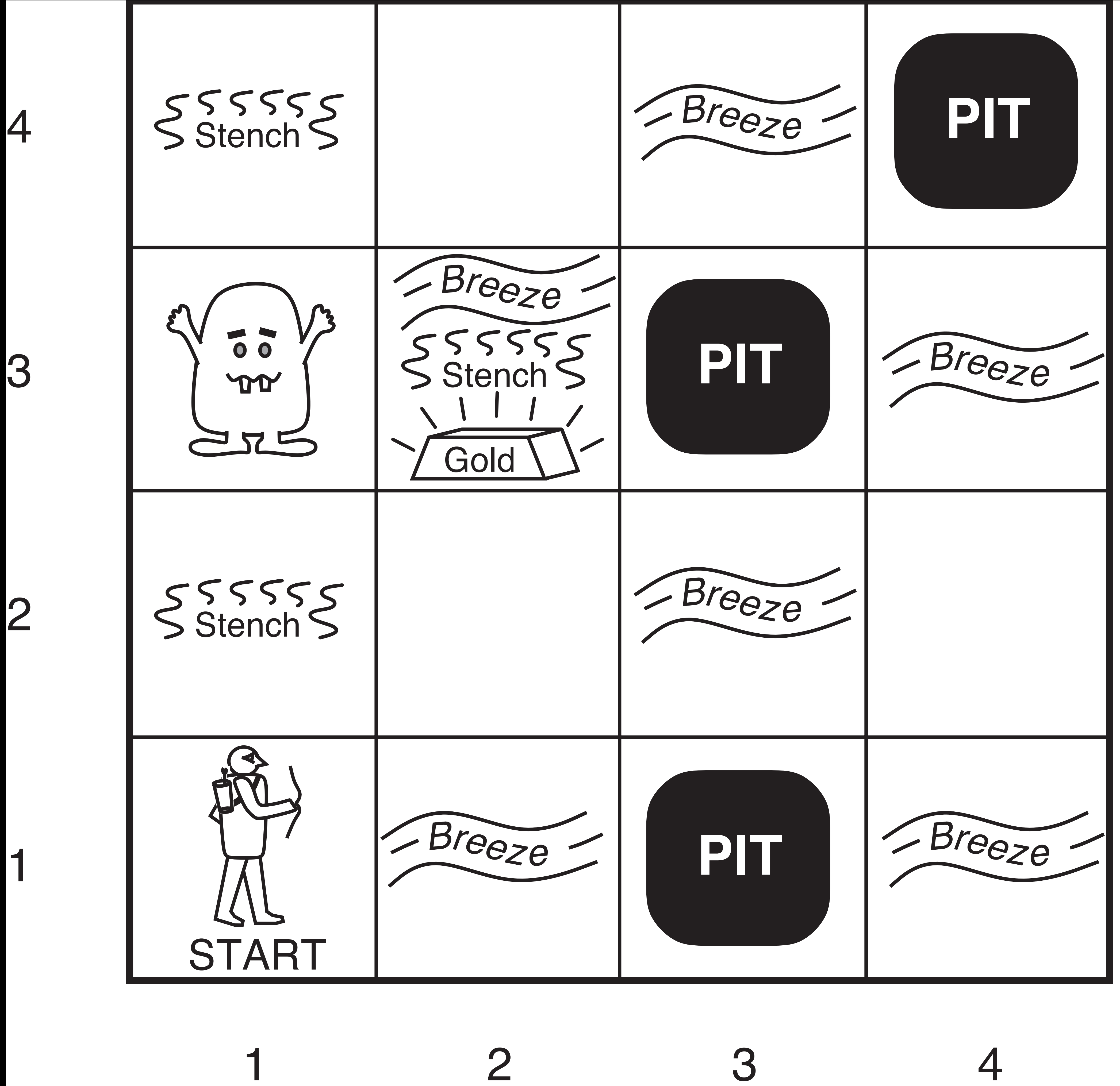
For a sound inference rule:

if $\alpha \vdash \beta$ then $\alpha \models \beta$

if $\alpha \vdash \gamma$ and $\gamma \vdash \beta$ then $\alpha \models \beta$

Proof

- Sequence of sound inference rule applications that lead from the premises to the desired conclusion


4
3
2
1
1
2
3
4
Stench
Breeze
PIT
Breeze
Stench
Gold
PIT
Breeze
Stench
Breeze
Breeze
PIT
Breeze
START

Background knowledge:

$$R_1 : \neg P_{1,1}$$

$$R_2 : B_{1,1} \Leftrightarrow (P_{1,2} \vee P_{2,1})$$

$$R_3 : B_{2,1} \Leftrightarrow (P_{1,1} \vee P_{2,2} \vee P_{3,1})$$

Perceptions:

$$R_4 : \neg B_{1,1}$$

$$R_5 : B_{2,1}$$

Biconditional elimination on R_2:

$$R_6 : ((B_{1,1} \Rightarrow (P_{1,2} \vee P_{2,1}) \wedge ((P_{1,2} \vee P_{2,1}) \Rightarrow B_{1,1})$$

And-elimination on R_6:

$$R_7 : (P_{1,2} \vee P_{2,1}) \Rightarrow B_{1,1}$$

Logical equivalence for contrapositives:

$$R_8 : \neg B_{1,1} \Rightarrow \neg (P_{1,2} \vee P_{2,1})$$

Modus Ponens on R_8 and R_4:

$$R_9 : \neg (P_{1,2} \vee P_{2,1})$$

DeMorgan's Rule:

$$R_{10} : \neg P_{1,2} \wedge \neg P_{2,1}$$

And-elimination on R_{10}:

$$R_{11} : \neg P_{1,2}$$

Propositional Inference As Search

- Initial state: Set of facts (initial knowledge base)
- Actions: Apply an inference rule to the sentences that match their premises
- Result: Add conclusions of inference rule to knowledge base
- Goal: The knowledge base contains the sentence we want to prove

Theorem Proving

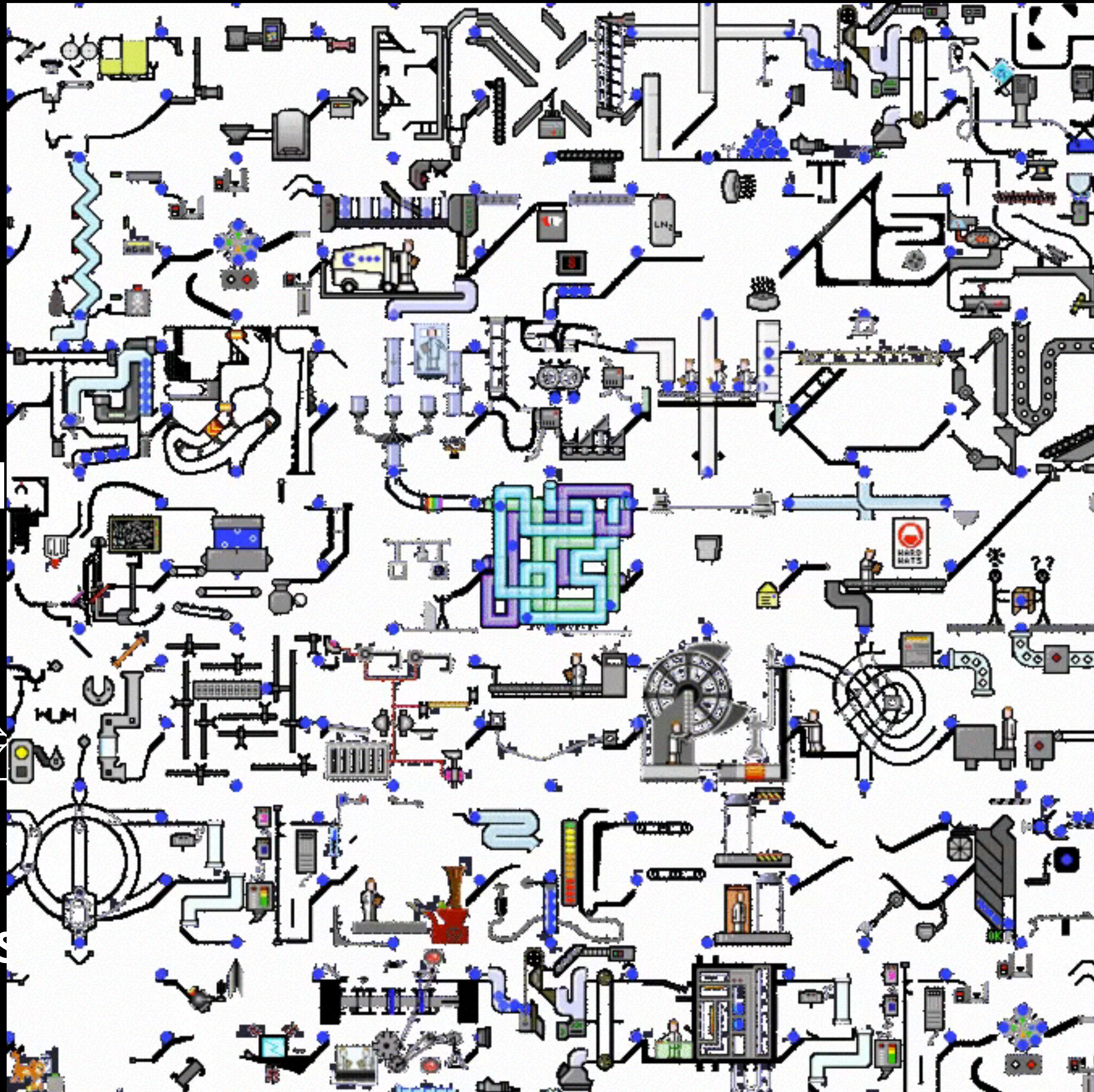
- Searching for proofs is an alternative to enumerating models
- *"In many practical cases, finding a proof can be more efficient because the proof can ignore irrelevant propositions, no matter how many of them there are."*

$$\frac{\ldots\alpha \vee \beta)}{\ldots \wedge \neg\beta}$$

And … n's

$$\frac{\alpha \Rightarrow \ldots}{\ldots} \qquad \frac{\ldots(\beta \Rightarrow \alpha)}{\beta}$$

Modus … al



There's Gotta Be a Simpler Way!

Literals

- Literal: propositional variable (P) or negation of propositional variable (¬Q)
- Complementary literals: one literal is the negation of another (P, ¬P)

Clauses

- Clause: disjunction of literals:

  $P \vee \neg Q \vee \neg R \vee \neg P$

- Unit clause: a single literal:

  P

  $\neg Q$

Unit Resolution

Clause

Unit Clause

$$\frac{l_1 \vee \cdots \vee l_i \vee \cdots \vee l_k, \quad m}{l_1 \vee \cdots \vee l_{i-1} \vee l_{i+1} \cdots \vee l_k}$$

Clause

$l_1, \ldots, l_k$ and m are literals

l_i and m are complementary

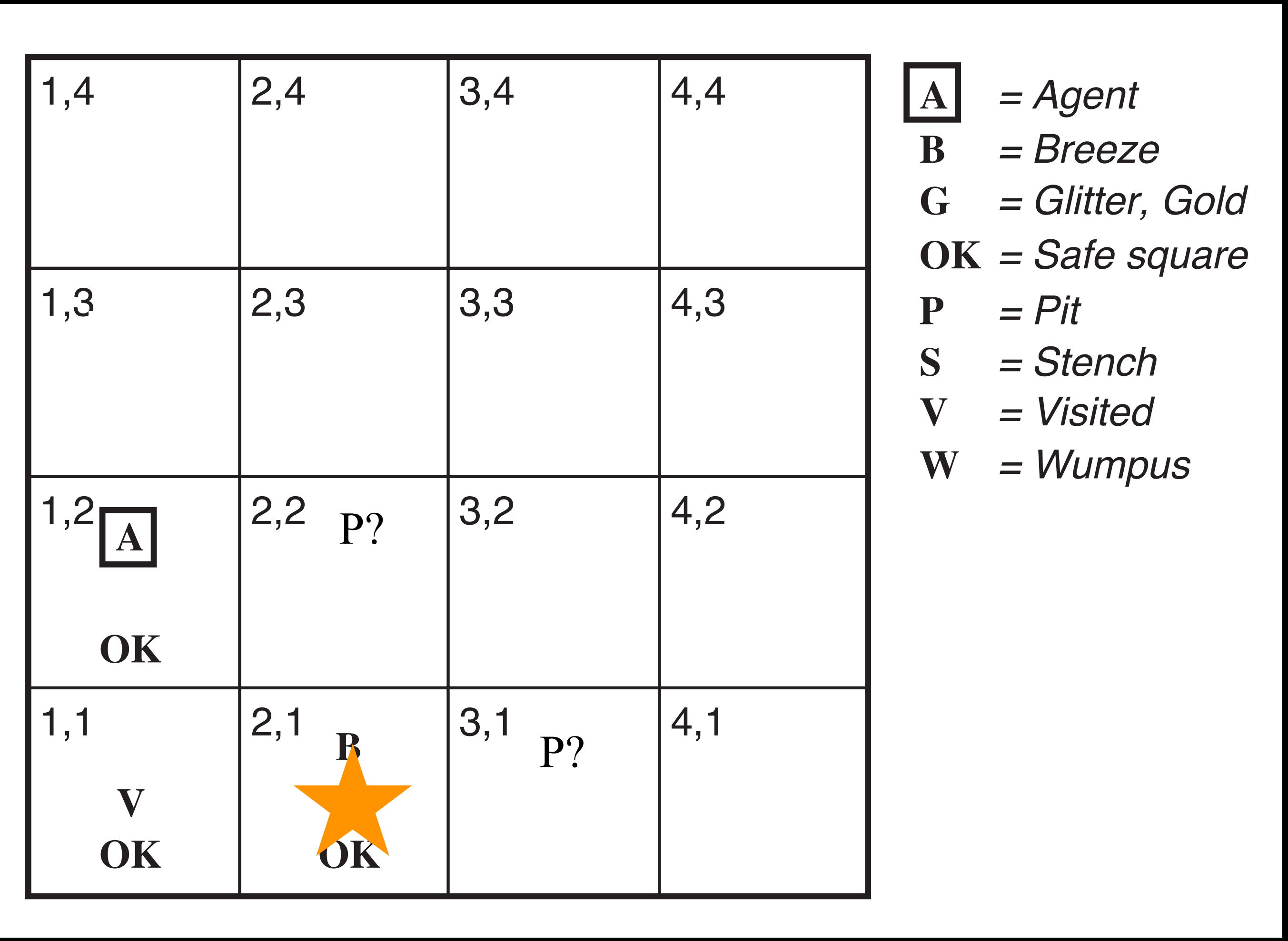
Complementary literals:

- Positive literal: P
- Negative literal: ¬P

$$Hungry \lor Cranky$$

$$\neg Hungry$$

Cranky



$$P_{1,1} \vee P_{2,2} \vee P_{3,1}$$

$$\neg P_{1,1}$$

$$P_{2,2} \vee P_{3,1}$$

$$\neg P_{2,2}$$

$$P_{3,1}$$

Unit Resolution

- Sound: if $\alpha \vdash \beta$ then $\alpha \models \beta$

- Not complete:

  if $\alpha \models \beta$ then $\alpha \vdash \beta$ ✗

Resolution

$$\frac{l_1 \vee \cdots \vee l_i \vee \cdots \vee l_k, \quad m_1 \vee \cdots \vee m_j \vee \cdots \vee m_n}{l_1 \vee \cdots \vee l_{i-1} \vee l_{i+1} \cdots \vee l_k \vee m_1 \vee \cdots \vee m_{j-1} \vee m_{j+1} \cdots \vee m_n}$$

$l_1, \ldots, l_k$, $m_1, \ldots, m_n$ are literals

l_i and m_j are complementary

Technical note: Resulting clause must be factored to contain only one copy of each literal.

$$\frac{P_{1,1} \lor P_{3,1}, \quad \neg P_{1,1} \lor \neg P_{2,2}}{P_{3,1} \lor \neg P_{2,2}}$$

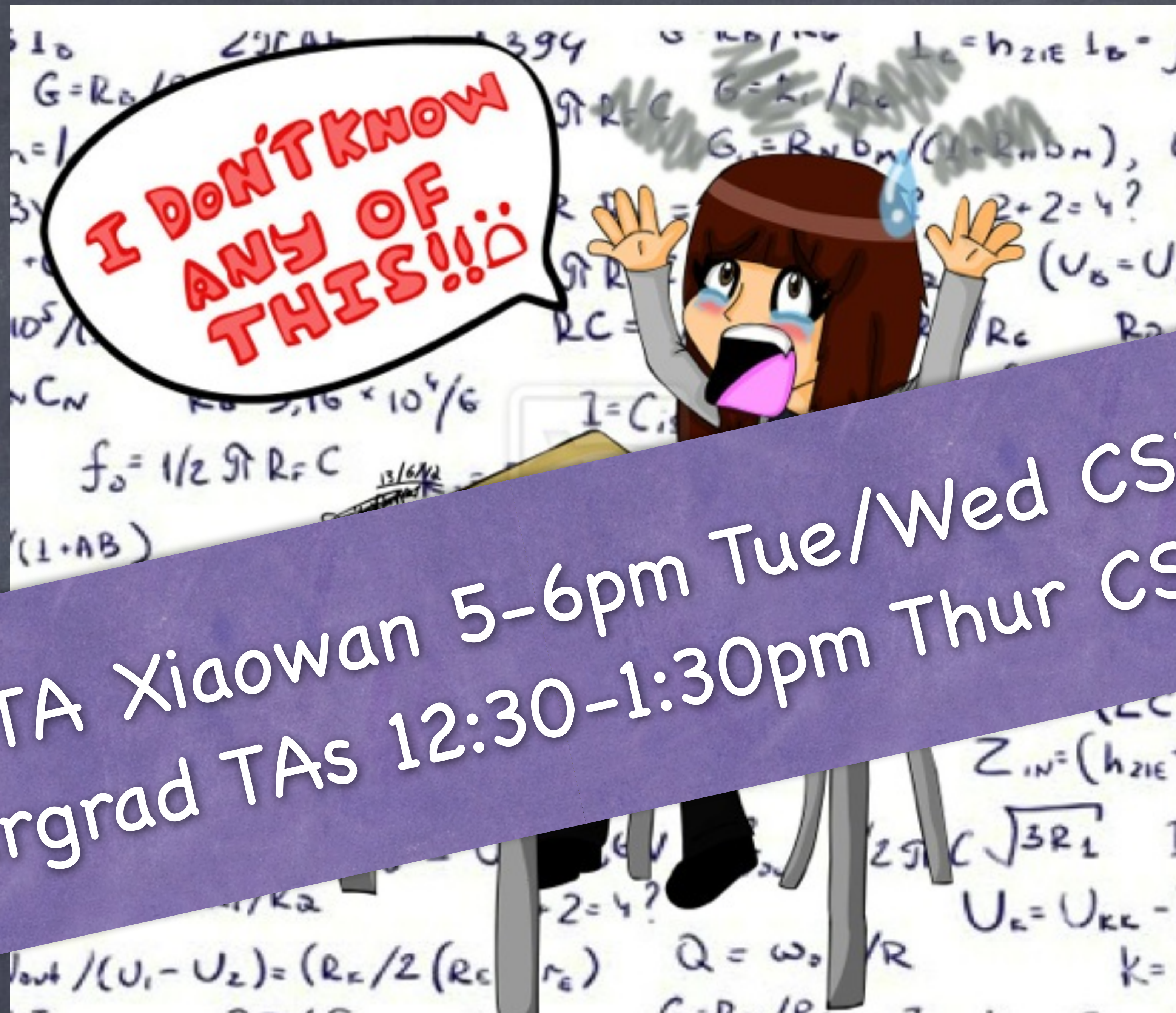
1. Hungry ∨ Cranky
2. ¬Sleepy ∨ ¬Hungry
3. Cranky ∨ Sleepy
4. ¬Sleepy ∨ Cranky (1,2)
5. Cranky ∨ Cranky (3,4)
6. Cranky (factoring)

Resolution

- Sound:
  - Easy to show if $\alpha \vdash \beta$ then $\alpha \vDash \beta$
- Complete?
  - "As good as complete"
  - if $\alpha \vDash \beta$ then $\alpha \vdash \beta'$ where β' is the same as β or is a shorter version of β
  - E.g. if $\alpha \vDash P \vee Q$ then

$$\alpha \vdash P \vee Q \quad \text{or} \quad \alpha \vdash P \quad \text{or} \quad \alpha \vdash Q$$

TO BE CONTINUED....



Grad TA Xiaowan 5–6pm Tue/Wed CSB 724
Undergrad TAs 12:30–1:30pm Thur CSB 633

DON'T LET THIS BE YOU! STUDY THE HOMEWORK SOLUTIONS! REWORK PROBLEMS YOU MISSED!

Conjunctive Normal Form (CNF)

- Eliminate ⇔: $\alpha \Leftrightarrow \beta \rightarrow \alpha \Rightarrow \beta \wedge \beta \Rightarrow \alpha$
- Eliminate ⇒: $\alpha \Rightarrow \beta \rightarrow \neg \alpha \vee \beta$
- Move negation in:
  - $\neg\neg\alpha \rightarrow \alpha$
  - $\neg(\alpha \vee \beta) \rightarrow (\neg\alpha \wedge \neg\beta)$
  - $\neg(\alpha \wedge \beta) \rightarrow (\neg\alpha \vee \neg\beta)$
- Distribute ∨ over ∧:
  - $(\alpha \vee (\beta \wedge \gamma)) \rightarrow ((\alpha \vee \beta) \wedge (\alpha \vee \gamma))$

$A \wedge B$ $\quad$ $P \Rightarrow Q$ $\quad$ $(A \vee B) \Rightarrow \neg(B \vee C)$

$A \wedge B$ $\quad$ $\neg P \vee Q$ $\quad$ $\neg(A \vee B) \vee \neg(B \vee C)$

A, B $\quad$ $\neg P \vee Q$ $\quad$ $(\neg A \wedge \neg B) \vee (\neg B \wedge \neg C)$

$$(\neg A \vee \neg B) \wedge (\neg A \vee \neg C) \wedge (\neg B \vee \neg B) \wedge (\neg B \vee \neg C)$$

$$(\neg A \vee \neg B) \wedge (\neg A \vee \neg C) \wedge \neg B \wedge (\neg B \vee \neg C)$$

$$(\neg A \vee \neg B), (\neg A \vee \neg C), \neg B, (\neg B \vee \neg C)$$

Inference Using Resolution

- Convert sentences (KB) to CNF (set of clauses)
- Apply resolution inference rule to pairs of clauses with complementary literals
- Add resulting clause to set of clauses
- Until...

Proof by Contradiction

- $\alpha \models \beta$ if and only if $(\alpha \wedge \neg\beta)$ is unsatisfiable
- If negation of goal is inconsistent with our knowledge
- Then the goal itself is entailed by our knowledge

Resolution Refutation

- Convert (KB $\wedge$ $\neg\alpha$) to CNF
- Apply resolution rule until:
  - No new clauses can be added

    ☛ KB does not entail α
  - Two clauses resolve to yield the empty clause

    ☛ KB entails α

Proof Using Resolution

- Proof by contradiction
- Derive empty clause from $(KB \wedge \neg\alpha)$ (converted to CNF, of course)

Effective Resolution

- Definite clauses
  - Disjunction of literals with <u>exactly</u> one positive literal
- Horn clauses
  - Disjunction of literals with <u>at most</u> one positive literal
- Natural reading as facts and “if-then” rules

Forward Chaining

- Knowledge base of definite clauses: facts and rules
- If premises of a rule (conjunction of literals) are known
  - Add its conclusion (single literal) to set of known facts
- Until either query is added or no further inferences can be made

Backward Chaining

- Work backward from query *q*
- If *q* is known to be true, we are done
- Otherwise find all rules whose conclusion (head) is *q*
  - If all the premises (body) of one of those rules can be proven true, then *q* is true

Effective Resolution

- Definite clauses
  - Disjunction of literals with exactly one positive literal
- Horn clauses
  - Disjunction of literals with at most one positive literal
- Natural reading as facts and “if-then” rules

Propositional Inference

- Entailment: "follows from our knowledge"
- Model checking
  - Intractable
  - But see also AIMA 7.6 & Project 2

Propositional Inference

- Inference rules: soundness, completeness
- Searching for proofs is an alternative to enumerating models
  - May be faster in practice

Propositional Inference

- Resolution is a sound and complete inference rule
  - Works on clauses (CNF)
- Special cases:
  - Definite & Horn clauses
  - Forward and backward chaining